COMPUTATIONAL FLUID DYNAMICS FOR AEROSPACE APPLICATIONS

MATHEMATICAL MODEL FOR TO FIND SUBSONIC & SUPERSONIC MACH NUMBER AT THE SPECIFIED AREA RATIO

ASSIGNMENT II

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In this article we are going to provide the Mathematical model to find the Mach number for the specified Area ratio at the subsonic & supersonic region.

Case 1:

Determination of subsonic Mach number;

The Area Mach number relation is given by;

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{\gamma-1}} \; ------\rightarrow (1)$$

Where A is the Cross sectional area of the stream tube;

A* is the Throat cross sectional area;

M Mach number;

$$\left(\frac{A}{A^*}M\right)^2 = \left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{\gamma-1}} -------\rightarrow (2)$$

Let;

$$E = \frac{\gamma+1}{\gamma-1}$$

Then equation (2) becomes,

$$\left(\frac{A}{A^*}M\right)^2 = \left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^E$$

$$\left(\frac{A}{A^*}M\right)^2 = \left(\frac{2}{\gamma+1}\,\frac{\gamma-1}{2}\left(\frac{2}{\gamma-1}+M^2\right)\right)^E$$

$$\left(\frac{A}{A^*}M\right)^2 = \left(\frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}+M^2\right)\right)^E$$

$$\left(\frac{A}{A^*}M\right)^2 = \left(\frac{1}{E}\left(\frac{2}{\gamma-1}+M^2\right)\right)^E$$

$$\left(\frac{A}{A^*}M\right)^2 = \left(\left(\frac{2}{\gamma+1}+\frac{\gamma-1}{\gamma+1}M^2\right)\right)^E \; ------\rightarrow (3)$$

Assume;

$$T = \left(\frac{A}{A^*}\right)^2, \qquad U = \frac{2}{\gamma + 1}, \qquad V = \frac{\gamma - 1}{\gamma + 1}$$

$$X = M^2$$

Then Equation (3) becomes,

$$TX = (U + VX)^E \quad ------\rightarrow (4)$$

$$F(X) = TX = (U + VX)^E ------\rightarrow (5)$$

$$F'(X) = EV(U + VX)^{E-1} ------\rightarrow (6)$$

$$EV = \frac{\gamma + 1}{\gamma - 1} \quad \frac{\gamma - 1}{\gamma + 1} = 1$$

$$U + V = \frac{2}{\gamma + 1} \quad \frac{\gamma - 1}{\gamma + 1} = 1$$

After substituting the above values equation (6) becomes

$$F'(X) = (U + VX)^{E-1}$$

$$F(X) = TX = (U + VX)^E$$

$$F(0) = U^E$$

$$F'(1) = (U + V)^{E-1} = 1$$

The general quadratic equation is

$$F(X) = a + bX + cX^2 ----\rightarrow (7)$$

$$F'(X) = b + 2cX$$

$$F(0) = a = U^E$$

$$F'(1) = b + 2C = 1$$

$$F(1) = a + b + c = 1$$

Substitute the values of b & c in terms of 'a' in equation (7), We get,

$$F(X) = a + (1-2a)X + aX^2 \qquad ------\rightarrow (8)$$

$$F(X) = TX = a + (1-2a)X + aX^2$$

$$aX^2 + (1-2a)X - TX + a = 0$$

$$X^2 + \frac{(1-T-2a)}{a}X + 1 = 0 \qquad -----\rightarrow (9)$$

The root value of the above equation provides the subsonic Mach number M_1 in terms of squared value.

$$X = {M_1}^2 = \frac{-\frac{(1-T-2a)}{a} - \sqrt{\frac{(1-T-2a)^2}{a^2} - 4}}{2} \qquad ------\rightarrow (10)$$

Finally the required subsonic Mach number

$$M_1 = \left(\frac{-\frac{(1-T-2a)}{a} - \sqrt{\frac{(1-T-2a)^2}{a^2} - 4}}{2}\right)^{\frac{1}{2}} \qquad ------\rightarrow (11)$$

Case 2:

Determination of supersonic Mach number;

We know that;

$$TX = (U + VX)^E$$

Take;

$$X = \frac{1}{Y}$$

Substitute the value of X in the above equation

We get;

$$\frac{T}{Y} = \left(U + \frac{V}{Y}\right)^E -------\rightarrow (11)$$

$$TY^{E-1} = (UY + V)^E$$

$$T^{\frac{1}{E-1}}\ Y = (UY + V)^{\frac{E}{E-1}} -----\rightarrow (12)$$

Let we take,

$$Z = T^{\frac{1}{E-1}} = \left(\frac{A}{A^*}\right)^{\frac{2}{E-1}}$$

Then equation (12) becomes,

$$Z\,Y = (UY+V)^{\frac{E}{E-1}} \qquad - - - - - \rightarrow (13)$$

$$F(Y) = (UY+V)^{\frac{E}{E-1}} \qquad - - - - - - \rightarrow (14)$$

$$F'(Y) = \frac{E}{E-1}U(UY+V)^{\frac{1}{E-1}} \qquad - - - - - - \rightarrow (15)$$

We know that,

$$\frac{E}{E-1}U = 1$$

Then,

$$F'(Y) = (UY+V)^{\frac{1}{E-1}} \qquad - - - - - \rightarrow (16)$$

$$F(0) = V^{\frac{E}{E-1}}$$

$$F(1) = (U+V)^{\frac{E}{E-1}} = 1$$

$$F'(1) = (U+V)^{\frac{1}{E-1}} = 1$$

The general quadratic equation is

$$F(Y) = a + bY + cY^2 \qquad - - - - - \rightarrow (17)$$

$$F(0) = a = V^{\frac{E}{E-1}}$$

$$F(1) = a + b + c = 1$$

$$F'(1) = b + 2c = 1$$

$$F(Y) = ZY = a + bY + cY^2 \qquad - - - - - - \rightarrow (18)$$

Substitute the values of b & c in equation (18) in terms of 'a' we get,

$$ZY = a + (1-2a)Y + aY^2 \quad - - - - \rightarrow (19)$$

$$aY^2 + (1-2a)Y - ZY + a = 0$$

$$Y^2 + \frac{(1-Z-2a)}{a}Y + 1 = 0 \quad - - - - - - - \rightarrow (20)$$

The root value of the above equation provides the supersonic Mach number M_2 in term of squared value.

$$Y = {M_2}^2 = \frac{-\frac{(1-Z-2a)}{a} + \sqrt{\frac{(1-Z-2a)^2}{a^2} - 4}}{2} \quad - - - - - - - \rightarrow (21)$$

Finally the required supersonic Mach number

$$M_2 = \left(\frac{-\frac{(1-Z-2a)}{a} + \sqrt{\frac{(1-Z-2a)^2}{a^2} - 4}}{2}\right)^{\frac{1}{2}} \quad - - - - - - - \rightarrow (22)$$